AN ALMOST EVERYWHERE EXISTENCE THEOREM FOR SOLUTIONS OF VOLTERRA FUNCTIONAL EQUATIONS(1)

BY

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1. **Introduction.** Let C_w be the Wiener space, i.e., the collection of real valued functions $x(t)$ defined and continuous on $I: 0 \leqq t \leqq 1$ and satisfying $x(0)=0$.

Let a finite sequence of real valued continuous functions $F^1(t, u)$, $F^2(t, u, v_1), \cdots, F^n(t, u, v_1, \cdots, v_{n-1})$ be defined and continuous for $t \in I$ and other variables unrestricted. The Volterra functionals $\Phi^k(x|t)$, $\Lambda^k(x|t)$ depending on the function $x(\cdot)$ and the real variable t are defined inductively by

$$\Lambda^0(x \mid t) = x(t), \qquad \text{on } C_w \otimes I, \tag{1}$$

$$\Phi^k(x \mid t) = F^k(t, \Lambda^0, \cdots, \Lambda^{k-1}), \qquad (k = 1, 2, \cdots, n) \text{ on } C_w \otimes I, \tag{2}$$

$$\Lambda^k(x \mid t) = \int_0^t \Phi^k(\tau)d\tau \qquad (k = 1, 2, \cdots, n) \text{ on } C_w \otimes I. \tag{3}$$

For any $x \in C_w$, the function y defined by the Volterra functional equation

$$y(t) = x(t) + \Lambda^n(x \mid t) \tag{4}$$

or with $f = F^n$

$$y(t) = x(t) + \int_0^t f[s, \Lambda^0(x \mid s), \cdots, \Lambda^{n-1}(x \mid s)]ds \tag{5}$$

belongs to C_w. In [3] we showed that under certain conditions on F^k there exists uniquely $x \in C_w$ satisfying (5) for every given $y \in C_w$. In the present article we prove an almost everywhere existence theorem for solutions of (5) where the phrase almost everywhere refers to the Wiener measure defined on C_w. Our result is the following:

THEOREM. *Let $F^1(t, u)$, $F^2(t, u, v_1), \cdots, F^{n-1}(t, u, v_1, \cdots, v_{n-2})$, $f(t, u, v_1, \cdots, v_{n-1})$ be continuous and have continuous first derivatives with respect to $u, v_1, \cdots, v_{n-1}$ on $I \otimes R_k$ $(k=1, 2, \cdots, n)$ where R_k is the k-dimensional Euclidean space and let f_t be continuous on $I \otimes R_n$. Let $F^1, F^2, \cdots, F^{n-1}$,*

Received by the editors August 21, 1961.

(1) The author is indebted to Professor R. H. Cameron for his advice in writing this paper.

f satisfy the order of growth conditions([2])

$$(6)\quad |F^k(t, u, v_1, \cdots, v_{k-1})| \leqq C \sum_{j=0}^{k-1} |v_j| \quad on\ I \otimes R_k\ (k = 1, 2, \cdots, n-1),$$

$$(7)\quad f(t, u, v_1, \cdots, v_{n-1}) \operatorname{sgn} u \geqq -A_1\{Bu\}^2 \qquad on\ I \otimes R_n,$$

$$(8)\quad f_u + 4g_t + 4\sum_{j=1}^{n-1} g_j F^j \leqq 2\sum_{j=0}^{n-1} \alpha_j^2 v_j^2 + A_2 \qquad on\ I \otimes R_n,$$

$$(9)\quad g(1, u, v_1, \cdots, v_{n-1}) \geqq -\frac{1}{2}\alpha(\cot\beta)u^2 - A_3 \qquad on\ R_n,$$

where

$$(10)\quad g(t, u, v_1, \cdots, v_{n-1}) = \int_0^u f(t, u', v_1, \cdots, v_{n-1})du' \qquad on\ I \otimes R_n$$

$$(11)\quad \alpha = \left\{\alpha_0^2 + \sum_{j=1}^{n-1} [C(1+C)^{j-1}]^2 \alpha_j^2\right\}^{1/2}$$

and A_1, A_2, B, C, α_j $(j=0, 1, \cdots, n-1)$, β *are positive constants satisfying* $\alpha<\beta<\pi$ *and* $B<1$. *Then corresponding to almost every* $y\in C_w$, (5) *has a solution* $x\in C_w$ *which is unique in* C_w.

2. **Lemma.** *Let each of* $F^1(t, u)$, $F^2(t, u, v_1)$, $\cdots$ *be continuous and satisfy* (6) *on* $I\otimes R_k$, $(k=1, 2, \cdots)$. *Then for any* $x\in C_w$ *the Volterra functional* $\Lambda^k(x|t)$, $(k=0, 1, 2, \cdots)$ *defined by* (1), (2), (3) *satisfies*

$$(12)\quad \Lambda^k(x\,|\,t) \leqq C(1+C)^{k-1}\left\{\int_0^t [x(s)]^2 ds\right\}^{1/2}$$

for $t\in I$, $(k=1, 2, \cdots)$.

Proof. The proof is based on Schwarz's inequality and a complete induction on k. For $k=1$, by (6) and Schwarz's inequality

$$\begin{aligned} |\Lambda^1(x\,|\,t)| &\leqq \int_0^t |F^1[s, x(s)]|\,ds \\ &\leqq C\int_0^t |x(s)|\,ds \\ &\leqq C\left\{\int_0^t [x(s)]^2 ds\right\}^{1/2} \qquad \text{for } t\in I. \end{aligned}$$

Now suppose that (12) holds for $1\leqq k\leqq N$. Then again by (6) and Schwarz's inequality

([2]) $v_0=u$.

$$
\begin{aligned}
|\Lambda^{N+1}(x\,|\,t)| &\leqq \int_0^t |F^{N+1}[s, x(s), \Lambda^1(x\,|\,s), \cdots, \Lambda^N(x\,|\,s)]|\,ds \\
&\leqq C\int_0^t |x(s)|\,ds + C\sum_{j=1}^{N}\int_0^t |\Lambda^j(x\,|\,s)|\,ds \\
&\leqq C\left\{\int_0^t [x(s)]^2 ds\right\}^{1/2} \\
&\quad + C\sum_{j=1}^{N}\int_0^t C(1+C)^{j-1}\left\{\int_0^s [x(r)]^2 dr\right\}^{1/2} ds \\
&\leqq C\left\{1 + C\sum_{j=1}^{N}(1+C)^{j-1}\right\}\left\{\int_0^t [x(s)]^2 ds\right\}^{1/2} \\
&= C(1+C)^N\left\{\int_0^t [x(s)]^2 ds\right\}^{1/2}
\end{aligned}
$$

and (12) holds for $k=N+1$ as well. This completes the proof of (12) by induction.

3. **Proof of the theorem.** Let $\gamma = B^{-1}-1>0$ and

(13) $$\phi(t, u) = (t+\gamma)^{-1/2}\exp\{(t+\gamma)^{-1}u^2\} \qquad \text{on } I\otimes R_1.$$

Since $t+\gamma>0$ on I, $\phi(t, u)$ has continuous derivatives of all orders with respect to t and u on $I\otimes R_1$. Define a function $G(t, u, v_1, \cdots, v_{n-1}, \lambda)$ depending on a non-negative parameter λ by

(14) $$G(t, u, v_1, \cdots, v_{n-1}\,|\,\lambda) = g(t, u, v_1, \cdots, v_{n-1}) + \lambda\phi(t, u) \qquad \text{on } I\otimes R_n,\ \lambda \geqq 0.$$

Then

(15) $$G_t = g_t - \frac{1}{2}\lambda(t+\gamma)^{-1}\phi(t, u) - \lambda(t+\gamma)^{-2}u^2\phi(t, u), \qquad \text{on } I\otimes R_n,\ \lambda \geqq 0,$$

(16) $$G_u = f + 2\lambda(t+\gamma)^{-1}u\phi(t, u), \qquad \text{on } I\otimes R_n,\ \lambda \geqq 0,$$

(17) $$G_j = g_j,\ (j = 1, 2, \cdots, n-1) \qquad \text{on } I\otimes R_n,\ \lambda \geqq 0,$$

and these derivatives are all continuous. Furthermore G_u has continuous first derivatives with respect to $t, u, v_1, \cdots, v_{n-1}$ on $I\otimes R_n$ for $\lambda\geqq 0$ and in particular G_{uu} is given by

(18) $$G_{uu} = f_u + 2\lambda(t+\gamma)^{-1}\phi(t, u) + 4\lambda(t+\gamma)^{-2}u^2\phi(t, u)$$

so that by (15)

(19) $$G_{uu} + 4G_t = f_u + 4g_t \qquad \text{on } I\otimes R_n \text{ for } \lambda \geqq 0.$$

Now $F^1, F^2, \cdots, F^{n-1}, G_u$ satisfy the conditions on $F^1, F^2, \cdots, F^{n-1}, f$ of Theorem 1 of [2] so that the transformation

$$y(t) = x(t) + \int_0^t G_u[s, \Lambda^0(x \mid s), \cdots, \Lambda^{n-1}(x \mid s) \mid \lambda]ds, \qquad \lambda \geqq 0, \tag{20}$$

transforms C_w in a 1-1 manner into a measurable subset Γ with a measure given by(3)

$$m_w(\Gamma) = \int_{C_w} \exp\{J(x, \lambda)\} d_w x, \qquad \lambda \geqq 0 \tag{21}$$

where

$$\begin{aligned} J(x, \lambda) = \int_0^1 & K[t, \Lambda^0(x \mid t), \cdots, \Lambda^{n-1}(x \mid t) \mid \lambda]dt \\ & + 2G(0, 0, \cdots, 0 \mid \lambda) - 2G[1, \Lambda^0(x \mid 1), \cdots, \Lambda^{n-1}(x \mid 1) \mid \lambda] \end{aligned} \tag{22}$$

for $x \in C_w$, $\lambda \geqq 0$ with

$$\begin{aligned} K(t, u, v_1, \cdots v_{n-1} \mid \lambda) &= \frac{1}{2} G_{uu} + 2G_t - G_u^2 + 2 \sum_{j=1}^{n-1} G_j F^j \\ &= \frac{1}{2} f_u + 2g_t - G_u^2 + 2 \sum_{j=1}^{n-1} g_j F^j, \qquad \text{for } \lambda \geqq 0 \end{aligned} \tag{23}$$

according to (19), (17).

We show next that for each positive value of λ the transformation (20) transforms C_w 1-1 onto itself. We only have to show that when $\lambda > 0$, G_u satisfies the condition (4.1) of Theorem II of [3]. From(16)

$$G_u \operatorname{sgn} u = f \operatorname{sgn} u + 2\lambda(t + \gamma)^{-1} \mid u \mid \phi(t, u) \qquad \text{on } I \otimes R_n \text{ for } \lambda \geqq 0 \tag{24}$$

and from (13) and $0 < t + \gamma \leqq 1 + \gamma$, $(t+\gamma)^{-1} \geqq B$ for $t \in I$,

$$\begin{aligned} 2\lambda(t + \gamma)^{-1} \mid u \mid \phi(t, u) &= 2\lambda(t + \gamma)^{-3/2} \mid u \mid \exp\{(t + \gamma)^{-1}u^2\} \\ &\geqq 2\lambda B^{3/2} \mid u \mid \exp\{Bu^2\}, \qquad \text{on } I \otimes R_1 \text{ for } \lambda \geqq 0. \end{aligned} \tag{25}$$

Now when $\lambda > 0$ and $|u| \geqq \lambda^{-1} B^{-3/2} A_1$, (25) implies

$$2\lambda(t + \gamma)^{-1} \mid u \mid \phi(t, u) \geqq 2A_1 \exp\{Bu^2\} \qquad \text{on } I \otimes R_1 \tag{26}$$

so that according to (24), (7), (26)

$$G_u \operatorname{sgn} u \geqq A_1 \exp\{Bu^2\} \geqq A_1 \tag{27}$$

for $\lambda > 0$, $|u| \geqq \lambda^{-1} B^{-3/2} A_1$, $(t, v_1, v_2, \cdots, v_{n-1}) \in I \otimes R_{n-1}$. On the other hand when $\lambda > 0$ but $|u| \leqq A^{-1} B^{-3/2} A_1$,

(3) See the first equation on p. 152 of [2].

$$f \operatorname{sgn} u \geqq -A_1 \exp\{B\lambda^{-2}B^{-3}A_1^2\} = -A(\lambda)$$

according to (7) where by definition

$$A(\lambda) = A_1 \exp\{\lambda^{-2}B^{-2}A_1^2\} > 0$$

and hence by (24), (25)

$$G_u \operatorname{sgn} u \geqq -A(\lambda) \tag{28}$$

for $\lambda > 0$, $|u| \leqq \lambda^{-1}B^{-3/2}A_1$, $(t, v_1, v_2, \cdots, v_{n-1}) \in I \otimes R_{n-1}$.

Summarizing (27), (28) we obtain

$$G_u \operatorname{sgn} u \geqq -A(\lambda) \qquad \text{on } I \otimes R_n \text{ with } A(\lambda) > 0 \text{ for } \lambda > 0. \tag{29}$$

Thus for each $\lambda > 0$, G_u satisfies (4.1) of [3] and according to Theorem II of [3] the transformation (20) transforms C_w 1-1 onto itself. From (21) we have

$$1 = \int_{C_w} \exp\{J(x, \lambda)\} d_w x \qquad \text{for } \lambda > 0. \tag{30}$$

Now since $G_u(t, u, v_1, \cdots, v_{n-1} | 0) = f(t, u, v_1, \cdots, v_{n-1})$ according to (16), we only have to show that (30) holds even when $\lambda = 0$ in order to complete the proof of the theorem. We show

$$\lim_{\lambda \downarrow 0} \int_{C_w} \exp\{J(x, \lambda)\} d_w x = \int_{C_w} \exp\{J(x, 0)\} d_w x. \tag{31}$$

This is done in what follows by interchanging the order of integration and limiting process.

According to (22), (14), for each $x \in C_w$

$$\begin{aligned} \lim_{\lambda \downarrow 0} J(x, \lambda) &= \lim_{\lambda \downarrow 0} \left\{ \int_0^1 K[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t) | \lambda] dt \right\} \\ &\quad - 2g(0, 0, \cdots, 0) - 2g[1, \Lambda^0(x | 1), \cdots, \Lambda^{n-1}(x | 1)]. \end{aligned} \tag{32}$$

To pass to the limit under the integral sign in (32) we show that for each fixed $x \in C_w$, K is bounded on I for $0 < \lambda \leqq 1$. From (23), (16), (13)

$$\begin{aligned} &K[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t) | \lambda] \\ &= \frac{1}{2} f_u[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t)] + 2g_t[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t)] \\ &- \{f[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t)] + 2\lambda(t + \gamma)^{-3/2}\Lambda^0(x | t) \\ &\qquad \cdot \exp\{(t + \gamma)^{-1}[\Lambda^0(x | t)]^2\}\}^2 \\ &+ 2\sum_{j=1}^{n-1} g_j[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t)] F^j[t, \Lambda^0(x | t), \cdots, \Lambda^{n-1}(x | t)] \\ &\qquad \text{for } t \in I, \lambda \geqq 0. \end{aligned} \tag{33}$$

From the continuity of $F^1, F^2, \cdots, F^{n-1}, f, f_u, g_t, g_j$ $(j=1, 2, \cdots, n-1)$ it is evident that K is bounded on I for $0<\lambda\leqq 1$ for each $x\in C_w$. Also from (23), (16)

$$\lim_{\lambda\downarrow 0} K(t, u, v_1, \cdots, v_{n-1} \mid \lambda) = \frac{1}{2} f_u + 2g_t - f^2 + 2\sum_{j=1}^{n-1} g_j F^j = K(t, u, v_1, \cdots, v_{n-1} \mid 0)$$

and from (32), (22), (14)

$$\lim_{\lambda\downarrow 0} J(x, \lambda) = \int_0^1 K[t, \Lambda^0(x \mid t), \cdots, \Lambda^{n-1}(x \mid t) \mid 0]dt - 2g(0, 0, \cdots, 0) - 2g[1, \Lambda^0(x \mid 1), \cdots, \Lambda^{n-1}(x \mid 1)] = J(x, 0). \tag{34}$$

We next justify

$$\lim_{\lambda\downarrow 0} \int_{C_w} \exp\{J(x, \lambda)\} d_w x = \int_{C_w} \lim_{\lambda\downarrow 0} \exp\{J(x, \lambda)\} d_w x \tag{35}$$

by dominating $\exp\{J(x, \lambda)\}$ on C_w for all $\lambda>0$ by a function which is independent of λ and integrable on C_w. From (22), (23), (14), (13)

$$\begin{aligned} J(x, \lambda) \leqq & \int_0^1 \Big\{\frac{1}{2} f_u[t, \Lambda^0, \cdots, \Lambda^{n-1}] + 2g_t[t, \Lambda^0, \cdots, \Lambda^{n-1}] \\ & + 2\sum_{j=1}^{n-1} g_j[t, \Lambda^0, \cdots, \Lambda^{n-1}] F^j[t, \Lambda^0, \cdots, \Lambda^{j-1}]\Big\} dt \\ & - \int_0^1 \{G_u[t, \Lambda^0, \cdots, \Lambda^{n-1} \mid \lambda]\}^2 dt \\ & + 2\{g(0, 0, \cdots, 0) + \lambda\gamma^{-1/2}\} \\ & - 2\{g[1, \Lambda^0(1), \cdots, \Lambda^{n-1}(1)] + \lambda B^{1/2} \exp\{B[\Lambda^0(1)]^2\}\}. \end{aligned}$$

The second integral in the right-hand side is non-negative. Also $g(0, 0, \cdots, 0) = 0$ by (10), and $\lambda B^{1/2} \exp\{B[\Lambda^0(1)]^2\} > 0$. Therefore when $1>\lambda>0$

$$\begin{aligned} J(x, \lambda) \leqq & \int_0^1 \Big\{\frac{1}{2} f_u[t, \Lambda^0, \cdots, \Lambda^{n-1}] + 2g_t[t, \Lambda^0, \cdots, \Lambda^{n-1}] \\ & + 2\sum_{j=1}^{n-1} g_j[t, \Lambda^0, \cdots, \Lambda^{n-1}] F^j[t, \Lambda^0, \cdots, \Lambda^{j-1}]\Big\} dt \\ & - 2g[1, \Lambda^0(1), \cdots, \Lambda^{n-1}(1)] + 2\gamma^{-1/2} \qquad \text{for all } x \in C_w. \end{aligned} \tag{36}$$

The right-hand side of (36) is independent of λ. By (8), (9), Lemma, (11)

$$
\begin{aligned}
J(x,\lambda) &\leqq \int_0^1 \left\{ \sum_{j=0}^{n-1} \alpha_j^2 [\Lambda^j(x\mid t)]^2 + \frac{A_2}{2} \right\} dt + \alpha \cot \beta [\Lambda^0(x\mid 1)]^2 + 2A_3 + 2\gamma^{-1/2} \\
&\leqq \alpha_0^2 \int_0^1 [x(t)]^2 dt + \int_0^1 \sum_{j=1}^{n-1} \alpha_j^2 [C(1+C)^{j-1}]^2 \left\{ \int_0^t [x(s)]^2 ds \right\} dt \\
&\quad + \frac{A_2}{2} + \alpha \cot \beta [x(1)]^2 + 2A_3 + 2\gamma^{-1/2} \\
&\leqq \alpha^2 \left\{ \int_0^1 [x(t)]^2 dt \right\} + \frac{A_2}{2} + \alpha \cot \beta [x(1)]^2 + 2A_3 + 2\gamma^{-1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\exp\{J(x,\lambda)\} &\leqq \exp\left\{ \alpha^2 \int_0^1 [x(t)]^2 dt + \alpha \cot \beta [x(1)]^2 \right\} \\
&\quad \cdot \exp\left\{ \frac{A_2}{2} + 2A_3 + 2\gamma^{-1/2} \right\}.
\end{aligned} \tag{37}
$$

According to §2 of [1], the right-hand side of (37) is integrable on C_w. Thus (35) is valid, (31) is valid by (34) and (30) holds for $\lambda = 0$, which means that for almost every $y \in C_w$, (5) has a solution $x \in C_w$. Its uniqueness follows from Remark 1, §2 of [3].

4. **An example.** We give an example with $n = 2$ to which the present almost everywhere existence theorem is applicable but not the everywhere existence theorem, Theorem II of [3]. Let

$$
\begin{aligned}
F^1(t, u) &= \sin u, \\
f(t, u, v) &= \frac{1}{10} (u^2 \sin 2u + 2u \sin^2 u) \sin^2 v.
\end{aligned}
$$

Then

$$
\begin{aligned}
f_u(t, u, v) &= \frac{1}{10} (2u^2 \cos 2u + 4u \sin 2u + 2 \sin^2 u) \sin^2 v, \\
g(t, u, v) &= \frac{1}{10} u^2 \sin^2 u \sin^2 v, \\
g_t = 0,\ g_v F^1 &= \frac{1}{10} u^2 \sin^3 u \sin 2v,
\end{aligned}
$$

so that

$$|F^1(t, u)| \leqq |u|,$$

$$|f(t, u, v)| \leqq \frac{1}{10}(u^2 + 2|u|) \leqq \exp\left\{\frac{1}{2}u^2\right\},$$

$$f_u + 4g_t + 4g_v F^1$$

$$\leqq \frac{1}{10}(2u^2 + 4|u| + 2) + \frac{4}{10}u^2 \leqq \frac{1}{10}(6u^2 + 6) + \frac{4}{10}u^2 = u^2 + 1,$$

$$g(1, u, v) \geqq 0,$$

and the conditions in the theorem are satisfied with $A_1 = A_2 = 1$, $A_3 > 0$, $B = 1/2$, $C \geqq 1$, $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_0 = 1 < \beta < \pi$. On the other hand (4.1) of [3] is violated.

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